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Linear-solvability condition in the Saffman-Taylor problem

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A reexamination of part of the mathematical framework of linear-solvability theory in the context of Hele-Shaw flow is made. The WKB solution of the problem is extended to include the infinite series of the expansion, instead of the first two terms as was originally done. The prefactor of the cusp function is found to change by about 5%. This is to be compared with the local geometrical model for dendritic growth, where the extended theory predicts a dramatic improvement.

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I. INTRODUCTION

Interface dynamics far from equilibrium has been a focus of intense research in recent years [1–3]. This is due to the practical and theoretical importance of the problem to the understanding of material-related phenomena. Of particular theoretical interest is the pattern formation and selection problem where one wishes to predict the long-time steady-state pattern following an initial interfacial instability. Specific systems that have received much attention include two-phase flow in a Hele-Shaw cell [4], dendritic growth of a crystal from a supercooled melt [3], directional solidification of a binary alloy [2], and explosive crystallization [5–7] of an amorphous solid. Common to these problems is the existence of an unstable interface driven by an external force. This force may be provided by the differences in pressure, temperature, or concentration of matter on the two sides of the interface. Experimentally steady-state patterns of the interface shape have been observed in these systems, and it was a theoretical challenge to predict these specific patterns from the equations of motion in question. The difficulties come from the fact that a boundary condition for solving the equation of motion depends on time; i.e., one must solve a moving boundary problem that is nonlocal. A further difficulty is related to a small parameter of the problem, surface tension, which is associated with the highest derivative of the equation of motion [1]. Thus simple perturbative treatment fails.

It was not until quite recently that a theoretical understanding of the pattern formation and selection problem started to take shape. Independently, several groups [8] focused on the difficulties mentioned above and discovered that a singular perturbation theory combined with stability analysis might provide an answer to the selection problem. In particular, it was found that if the surface tension is treated correctly, then the continuously infinite possible solutions of the equation of motion in zero-surface-tension break into a discrete set where only one of the solutions is linearly stable to noise; thus only this one can survive and be selected by nature. This scenario, termed microscopic solvability, has been applied to several above-mentioned pattern-forming systems, and very interesting results have been obtained [1, 3]. It is now generally accepted that for the Hele-Shaw flow, this scenario provides a reasonable solution for interface patterns. For other systems, notably dendritic growth, the success of solvability theory depends on an additional small parameter, the anisotropy of the surface tension; thus some debate still exists in the literature [9]. However, it is reasonable to say that solvability theory provides our best understanding of the problem to date.

While qualitative results from solvability theory are consistent with those of observations, detailed comparisons are hard to make due to experimental difficulties in accurately determining system parameters. There is some success in comparing theory with numerical solutions of the equations of motion [10], but the latter are often hampered by numerical difficulties since the problem is inherently nonlinear and nonlocal. Even for simpler models, such as the geometrical model where nonlocality is completely neglected, disagreement has existed between the theory and numerics on the prefactors of the cusp function (see below). Thus it is desirable to carefully reexamine various mathematical approximations made in the original proposals of solvability theory and possibly improve it systematically.

The purpose of this paper is to provide a reexamination of a particular part of the mathematical framework of solvability theory in the context of the Hele-Shaw flow, where a full nonlocal model must be solved. The prefactor of the cusp function is recomputed to include all nonlinear terms at the level of a WKB approximation (see below), rather than just the leading term, as was done originally for simplicity. For the local geometrical model, Hakim [11] recently showed that such a procedure will dramatically improve the agreement between theory and numerical solution. For the Hele-Shaw-flow problem, a 5% numerical difference is found between this extended method and the original theory. The paper is organized as follows. In Sec. II, for completeness of the presentation, the fundamental equations of interest are set up and a brief review of solvability theory in connection to the prefactor of the cusp function is made. Section III is devoted to the calculation of the prefactor of the cusp function with this extended method, and Sec. IV is reserved for a short conclusion.

II. THE SAFFMAN-TAYLOR PROBLEM

The Saffman-Taylor problem [4] is the prediction of the steady-state shape of the fluid interface in a two-phase flow confined in a linear Hele-Shaw cell, where a less viscous fluid is pushing a more viscous one. It can be regarded as a model of, say, water pushing oil in a porous medium. The understanding of this problem is important, since it represents a class of pattern-forming systems where interfacial instabilities evolve and steady-state patterns are selected by nature.

The instability in the linear Hele-Shaw flow is the well-known Mullins-Sekerka instability [12], where the interfacial tension at the fluid interface is unable to stabilize long-wavelength fluctuations; thus the interface deforms from flat as the flow continues. Different modes grow and compete dynamically. The competition leads eventually to a single finger-shaped pattern in the Hele-Shaw cell at large times [13]. This shape is called a viscous finger. Experimentally [4, 14], the finger is characterized by its width λ in units of the channel width W. It is found that λ is usually greater than one-half of the channel width. Also, λ is a unique function of a control parameter γ, and approaches $\frac{1}{2}W$ as $\gamma \to 0$. The parameter γ is defined as

$$\gamma \equiv \frac{\tilde{\gamma}}{12\mu v_\infty}\left(\frac{b}{a}\right)^2, \tag{1}$$

where $\tilde{\gamma}$ is the interfacial tension, μ is the viscosity of the fluid that is being pushed, b is the gap spacing of the Hele-Shaw cell, $2a = W$ is the channel width, and v_∞ is the velocity of the fluid very far from the interface.

Theoretically, the governing equation of the system can be written from Darcy's law [4], which relates the flow velocity $\mathbf{v}$ with the pressure p,

$$\mathbf{v} = -\frac{b^2}{12\mu}\nabla p. \tag{2}$$

This and the assumption of incompressibility of fluids lead to the Laplace equation for the pressure,

$$\nabla^2 p = 0. \tag{3}$$

The conservation of matter at the interface and the Gibbs-Thompson relation provide the necessary boundary conditions,

$$v_n = -\frac{b^2}{12\mu}\nabla p \cdot \hat{\mathbf{n}} \tag{4}$$

and

$$p = -\tilde{\gamma}\kappa, \tag{5}$$

where $\hat{\mathbf{n}}$ is the normal to the interface and κ is the local curvature. The difficulty of solving these equations lies at the moving boundary condition (5), where κ depends on the solution of the problem, which is unknown. If we neglect the small parameter interfacial tension $\tilde{\gamma}$, the problem can then be solved in a closed form [1]. However, in that case one can only predict a combination of the finger width λ and the flow velocity v, not each separately; thus a continuous family of solutions exists. It can further be shown [1] that any finite perturbation treatment to include γ fails due to the fact that γ appears as the coefficient of the highest derivative in a different form of the equation of motion. A selection scenario is given by solvability theory, part of which we review briefly in connection with the prefactor of the cusp function.

Equations (3), (4), and (5) can be transformed into an integro-differential equation. After linearizing around the zero-surface-tension solution [15] and assuming a symmetric shape for the viscous finger, this equation can be written as

$$\nu\frac{d^2\Theta(\eta)}{d\eta^2} + Q_1(\eta)\Theta(\eta) + \frac{1}{\pi}\int_{-\infty}^{\infty} d\eta' \, \frac{Q_2(\eta,\eta')\Theta(\eta')}{\eta-\eta'} = R(\eta) \; , \tag{6}$$

where $\nu = \gamma\pi^2\frac{\lambda}{(1-\lambda)^2}$, $\Theta(\eta)$ is an antisymmetric function of its argument, and

$$Q_1(\eta) = \frac{4\beta^4(1+\eta^2)^{\frac{1}{2}}}{(1+\beta^2\eta^2)^2}, \tag{7}$$

$$Q_2(\eta,\eta') = \frac{4\eta\beta^4(1+\eta^2)^{\frac{1}{4}}(1+\eta'^2)^{\frac{1}{4}}}{(1+\beta^2\eta^2)^{\frac{1}{2}}(1+\beta^2\eta'^2)^{\frac{3}{2}}}, \tag{8}$$

$$R(\eta) = \frac{\eta[3+\beta^2(\eta^2-2)]}{(1+\beta^2\eta^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{9}{4}}}, \tag{9}$$

and

$$\beta = \frac{\lambda}{1-\lambda}.$$

In terms of Cartesian coordinates, η is the slope, which varies from $-\infty$ to ∞ as one goes all the way around the finger, passing through $\eta = 0$ at the tip of the finger. In terms of the angle of orientation of the interface, $\theta = \theta_0 + \nu\theta_1$, with θ_0 the zero-surface-tension solution of the problem,

$$\Theta(\eta) = \frac{(1+\beta^2\eta^2)^{1/2}}{(1+\eta^2)^{1/4}}\theta_1(\eta).$$

Equation (6) has the form

$$\mathcal{L}\Theta(\eta) = R(\eta). \tag{10}$$

Multiplying by a function Θ_0 and integrating leads, after some algebraic manipulation, to the following equation:

$$\int_{-\infty}^{0} d\eta\,\Theta_0(\eta)\mathcal{L}\Theta(\eta) = [\Theta_0\nu\Theta' - \Theta_0'\nu\Theta]|_{-\infty}^{0} + \int_{-\infty}^{0} d\eta\,\Theta(\eta)\nu\frac{d^2\Theta_0(\eta)}{d\eta^2} + \int_{-\infty}^{0} d\eta\Theta(\eta)Q_1(\eta)\Theta_0(\eta) + \int_{-\infty}^{0} d\eta\,\Theta(\eta)\frac{1}{\pi}\left\{\int_{-\infty}^{\infty} d\eta' \, \frac{Q_2(\eta',\eta)\Theta_0(\eta')}{\eta'-\eta}\right\}, \tag{11}$$

where $\Theta_0(-\eta) = -\Theta_0(\eta)$. We define an operator $\mathcal{L}^\dagger$ such that

$$\mathcal{L}^\dagger\Theta_0 \equiv \nu\frac{d^2\Theta_0(\eta)}{d\eta^2} + Q_1(\eta)\Theta_0(\eta) + \frac{1}{\pi}\int_{-\infty}^{\infty} d\eta' \, \frac{Q_2(\eta',\eta)\Theta_0(\eta')}{\eta'-\eta}. \tag{12}$$

Now, if we can find a solution to the equation

$$\mathcal{L}^\dagger\Theta_0 = 0, \tag{13}$$

all the integrals on the right-hand side of Eq. (11) will vanish. Using (10), we obtain

$$\int_{-\infty}^{0} d\eta\,\Theta_0\mathcal{L}\Theta = \int_{-\infty}^{0} d\eta\,\Theta_0 R(\eta) = [\Theta_0\nu\Theta' - \Theta_0'\nu\Theta]|_{-\infty}^{0}. \tag{14}$$

The cusp function is defined as

$$\Lambda \equiv \int_{-\infty}^{\infty} d\eta\,\Theta_0 R(\eta). \tag{15}$$

It is the vanishing of the cusp function that gives the condition for the selection of the pattern. Since $R(-\eta) = -R(\eta)$, Λ can be written as

$$\Lambda = 2[\Theta_0\nu\Theta' - \Theta_0'\nu\Theta]|_{-\infty}^{0}. \tag{16}$$

What one does next is to find a null eigenvector of the operator $\mathcal{L}^\dagger$, i.e., find a solution to (13). Suppose that the solutions of (13) have the WKB form $e^{\frac{S}{\sqrt{\nu}}}$, where the real part of $S < 0$, and S has points of stationary phase [i.e., points where $S'(\bar{\eta}) = 0$]. Then, in the limit of $\nu \to 0$, the integral can be evaluated by expanding the exponent around the point of stationary phase. The only contribution to the integral that is not exponentially small comes from the pole at $\eta' = \eta$, and the equation for the null eigenvector of $\mathcal{L}^\dagger$ becomes

$$\nu\frac{d^2\Theta_{0\pm}(\eta)}{d\eta^2} + Q_\pm\Theta_{0\pm}(\eta) = 0, \tag{17}$$

where

$$Q_\pm(\eta) = \frac{4\beta^4(1\pm i\eta)^{\frac{3}{2}}(1\mp i\eta)^{\frac{1}{2}}}{(1+\beta^2\eta^2)^2}. \tag{18}$$

The null eigenvector Θ_0 is given by the antisymmetric combination of $\Theta_{0\pm}$,

$$\Theta_0 = \frac{1}{2i}\left[\Theta_{0+} - \Theta_{0-}\right] = \mathrm{Im}\Theta_{0+}. \tag{19}$$

As mentioned above, one searches for a WKB solution, i.e., $\Theta_0 \sim \exp(S/\sqrt{\nu})$, where S can be expanded in powers of $\sqrt{\nu}$,

$$S = \sum_{n=0}^{\infty} S_n\nu^{\frac{n}{2}}. \tag{20}$$

In the formulation of Hong and Langer [15], only the first two terms of this expansion were included. That led to, for $\beta^2 < 1$,

$$\Lambda(\lambda,\nu) \simeq N\frac{(1-2\lambda)^{1/14}\lambda^{6/7}}{(1-\lambda)}\frac{1}{\nu^{13/28}}e^{\frac{E(\lambda)}{\sqrt{\nu}}}, \tag{21}$$

where $N = 2.008$ and

$$E(\lambda) = -2\beta^2\int_0^1 du\,\frac{(1-u)^{3/4}(1+u)^{1/4}}{1-\beta^2u^2}. \tag{22}$$

A point worth noting is that the prefactor of Λ, N, is not uniquely determined since one can multiply the null eigenvector of the operator $\mathcal{L}^\dagger$ by any quantity independent of η without changing Eq. (13). But since we are interested in this prefactor, proper normalization is needed. The prefactors of quantities such as $\frac{\Lambda}{\Theta_0|_{\rm tip}}$ or $\frac{\Lambda}{\Theta_0'|_{\rm tip}}$ are uniquely determined. Alternatively, the normalization of the null eigenvector or its derivative will make Λ uniquely determined. This was done by Hakim for the geometrical model [11].

As mentioned above, at the tip of the viscous finger we have $\Theta = \theta_1$, where θ_1 is the ν-dependent part of the exact solution of the shape, i.e., $\theta = \theta_0 + \nu\theta_1$, where θ_0 is the zero-surface-tension solution. Thus at the tip, $\theta + \frac{\pi}{2} = \nu\theta_1|_{\rm tip}$, which is given by the normalized cusp function,

$$\Lambda' \equiv \frac{\Lambda}{2\Theta_0'|_{\rm tip}} = -\nu\Theta|_{\rm tip} = -\nu\theta_1|_{\rm tip},$$

provided that we set $\Theta'|_{\rm tip} = 0$. Also, note that the term at $\eta \to -\infty$ in (16) vanishes [15]. Following Hong and Langer [15], $\Theta_0 = \mathrm{Im}\Theta_{0+}$ is given by

$$\Theta_{0+} = \frac{e^{\frac{S_0}{\sqrt{\nu}}}}{Q_+^{1/4}},$$

and since $S_0' = iQ_+^{1/2}$, $\Theta_{0+}'|_{\rm tip} = i\frac{\sqrt{2}\beta}{\sqrt{\nu}} +$ (nondominant terms). The normalized cusp function can be written as

$$\Lambda' = M\frac{(1-2\lambda)^{1/14}}{\lambda^{1/7}}\nu^{1/28}e^{\frac{E(\lambda)}{\sqrt{\nu}}}, \tag{23}$$

where $M = \frac{N}{2\sqrt{2}} = 0.71$.

In Sec. III, we shall recompute the prefactor M including all the terms in the WKB expansion (20). For the geometrical model of dendritic growth where no nonlocality is involved, Hakim recently showed that including all terms of the WKB series drastically improved the numerical value of the prefactor.

III. MATHEMATICAL REEXAMINATION

To include the whole series of the WKB expansion (20), we propose a solution for the null eigenvector of $\mathcal{L}^\dagger$ of the form

$$\Theta_0 = e^{\frac{S_0}{\sqrt{\nu}}}\sum_{n=0}^{\infty} g_n\nu^{\frac{n}{2}}. \tag{24}$$

This is equivalent to (20). In the following, we denote Θ_{0+} by Θ_0 and Q_+ by Q. Substituting this into (17), we obtain

$$(S_0'^2+Q)g_0+\ [(S_0'^2+Q)g_1+(2S_0'g_0'+S_0''g_0)]\nu^{\frac{1}{2}} + \sum_{m=0}^{\infty}\ [g_m''+(S_0'^2+Q)g_{m+2} +(2S_0'g_{m+1}'+S_0''g_{m+1})]\nu^{\frac{m}{2}+1}=0. \quad (25)$$

Equating terms with the same power in ν, we have

(a) $(S_0'^2+Q)g_0=0,$

(b) $(S_0'^2+Q)g_1+(2S_0'g_0'+S_0''g_0)=0,$

(c) $g_m''+(S_0'^2+Q)g_{m+2}+(2S_0'g_{m+1}'+S_0''g_{m+1})=0,$

which gives a set of equations for S_0, g_0, g_1, ..., g_n. From (a), we have $S_0'^2+Q=0$, so

$$S_0=i\int_0^{\eta}d\eta\,Q^{1/2}, \quad (26)$$

and

(b) $g_0'+\dfrac{S_0''}{2S_0'}g_0=0,$

(c) $g_{m+1}'+\dfrac{S_0''}{2S_0'}g_{m+1}=-\dfrac{1}{2S_0'}g_m''.$

Since S_0 has a point of stationary phase at $\bar{\eta}=i$, we evaluate our equations in the immediate neighborhood of $\bar{\eta}$. Let $\eta=i+\omega$, then

$$Q\simeq a\omega^{3/2},\quad \text{where } a\equiv\frac{4\beta^4 i^{3/2}2^{1/2}}{(1-\beta^2)^2},$$

$$\frac{S_0''}{2S_0'}=\frac{1}{4}\frac{Q'}{Q}\simeq\frac{3}{8}\omega^{-1}\text{ and }2S_0'=2ia^{1/2}\omega^{3/4}.$$

The equations around $\bar{\eta}$ become

(b) $g_0'+\dfrac{3}{8}\omega^{-1}g_0=0,$

(c) $g_{m+1}'+\dfrac{3}{8}\omega^{-1}g_{m+1}=-\dfrac{1}{2ia^{1/2}\omega^{3/4}}g_m''.$

From (b) we obtain

$$g_0=a_0\omega^{-3/8}, \quad (27)$$

and (c) gives a recursive relation for the rest of the g_n's. It is just a first-order equation of the form $g'+Fg=G$ whose solution is $g=g_h\int d\omega\,\frac{G}{g_h}$, where $g_h=e^{-\int d\omega\,F}$. The solution for g_{m+1} is then given in terms of g_m by

$$g_{m+1}=\omega^{-A}B\int d\omega\,\omega^{A-\frac{3}{4}}g_m'', \quad (28)$$

with $A=3/8$ and $B=-1/2ia^{1/2}$. It is clear that if g_m has the form $g_m=a_m\omega^{-A_m}$, g_{m+1} will have the same form. And since g_0 has the form $g_0=a_0\omega^{-A_0}$, with $A_0=3/8$, all we need to do is to find a recursive relation among the coefficients. When the form $g_m=a_m\omega^{-A_m}$ is put into (28), we find

$$g_{m+1}=\frac{BA_m(A_m+1)a_m}{A-A_m-\frac{7}{4}}\omega^{-(A_m+\frac{7}{4})}=a_{m+1}\omega^{-A_{m+1}}. \quad (29)$$

Comparison of the exponents of ω determines A_m

$$A_m=\frac{7}{4}m+A_0, \quad (30)$$

and comparison of the multiplicative coefficients determines a_m

$$a_m=\left(-\frac{7}{4}\right)^m B^m\frac{\Gamma(m+\frac{3}{14})\Gamma(m+\frac{11}{14})}{\Gamma(\frac{3}{14})\Gamma(\frac{11}{14})\Gamma(m+1)}a_0. \quad (31)$$

For consistency, we need to expand $e^{\frac{S_0}{\sqrt{\nu}}}$ and R around the point of stationary phase $\bar{\eta}$. Using Eqs. (18) and (26), S_0 can be approximated as

$$S_0\simeq E(\lambda)+\frac{8}{7}2^{1/4}i^{7/4}\frac{\lambda^2}{1-2\lambda}\omega^{7/4}, \quad (32)$$

where $\omega=\eta-i$ and $E(\lambda)$ is given by (22). We can write

$$e^{\frac{S_0}{\sqrt{\nu}}}=e^{\frac{E(\lambda)}{\sqrt{\nu}}}e^{\frac{\alpha\omega^{7/4}}{\sqrt{\nu}}},$$

where

$$\alpha\equiv\frac{8}{7}2^{1/4}i^{7/4}\frac{\lambda^2}{1-2\lambda}.$$

With the above results, (24) becomes

$$\Theta_0=e^{\frac{E(\lambda)}{\sqrt{\nu}}}\sum_{m=0}^{\infty}a_m\nu^{\frac{m}{2}}\omega^{-\frac{7}{4}m-\frac{3}{8}}e^{\frac{\alpha\omega^{7/4}}{\sqrt{\nu}}}, \quad (33)$$

and R, given by Eq. (9), becomes

$$R=b\,\omega^{-9/4},$$

where

$$b\equiv\frac{3\,i^{-5/4}}{2^{9/4}}(1-\beta^2)^{1/2}=\frac{3\,i^{-5/4}}{2^{9/4}}\frac{(1-2\lambda)^{1/2}}{(1-\lambda)}.$$

We now can write the cusp function defined in (15). Note that in this section Θ_0 denotes Θ_{0+}, thus

$$\Lambda=\frac{1}{i}\int_{-\infty}^{\infty}d\omega\,\Theta_0R(\eta)=\frac{1}{i}be^{\frac{E(\lambda)}{\sqrt{\nu}}}\sum_{m=0}^{\infty}a_m\nu^{\frac{m}{2}}I_m, \quad (34)$$

where I_m is defined as

$$I_m\equiv\int_{-\infty}^{\infty}d\omega\,\omega^{-\frac{7}{4}(m+\frac{3}{2})}e^{\frac{\alpha\omega^{7/4}}{\sqrt{\nu}}},$$

which can be written in terms of the inverse of a Γ function as

$$I_m=\frac{8\pi i}{7}\left(\frac{\alpha}{\sqrt{\nu}}\right)^{m+\frac{13}{14}}\frac{1}{\Gamma(m+\frac{27}{14})}.$$

The cusp function can then be written as

$$\Lambda = \left[\frac{8\pi}{7} a_0\, b\, \alpha^{13/14} \frac{1}{\nu^{13/28}}\right] e^{\frac{E(\lambda)}{\sqrt{\nu}}}\,\Delta,$$

where

$$\begin{aligned}\Delta &= \frac{1}{a_0}\sum_{m=0}^{\infty} \alpha^m a_m \frac{1}{\Gamma(m+\frac{27}{14})}\\ &= \frac{1}{\Gamma(\frac{3}{14})\Gamma(\frac{11}{14})}\sum_{m=0}^{\infty}\left[\frac{1}{2}\right]^m \frac{\Gamma(m+\frac{3}{14})\Gamma(m+\frac{11}{14})}{\Gamma(m+1)\Gamma(m+\frac{27}{14})}. \end{aligned} \quad (35)$$

Equation (31) has been used to write the second equality. a_0 is chosen to be consistent with Hong and Langer's calculation [15]. There is an overall constant that is irrelevant since what is uniquely determined is the normalized cusp function. To compare with the original calculation, we set a_0 to agree with the result of Hong and Langer (21) when only the term $m = 0$ in (35) is considered. We have

$$\Lambda = N\frac{(1-2\lambda)^{1/14}\lambda^{6/7}}{(1-\lambda)}\frac{1}{\nu^{13/28}} e^{\frac{E(\lambda)}{\sqrt{\nu}}}\,\Delta\Gamma\left(\frac{27}{14}\right).$$

where

$$N = 3\pi\frac{2^{22/7}}{7^{27/14}\Gamma(\frac{27}{14})} = 2.008.$$

Hence the quantity $\Delta\Gamma(\frac{27}{14})$ gives a multiplicative factor that tells us how different this result is from the result where only two terms in Eq. (20) are kept. So, when setting $m = 0$, $\Delta\Gamma(\frac{27}{14}) = 1$ and the result of Eq. (21) is recovered. Finally, the uniquely determined cusp function Λ' is

$$\Lambda' = N'\frac{(1-2\lambda)^{1/14}}{\lambda^{1/7}}\nu^{1/28} e^{\frac{E(\lambda)}{\sqrt{\nu}}}. \quad (36)$$

Here $N' = M\Delta\Gamma(\frac{27}{14})$. The factor Δ, given by (35), is computed numerically and $\Gamma(\frac{27}{14})\Delta = 1.054$ is found, so $N' = 0.748$. Thus including all terms in the WKB expansion leads to a 5% change in the prefactor.

IV. CONCLUSION

We have reexamined part of the mathematical framework of solvability theory in the context of Hele-Shaw-flow. The prefactor of the cusp function is recomputed to include all nonlinear terms for the null eigenvector of the operator $\mathcal{L}^\dagger$ (13), rather than just the leading term as done originally by Hong and Langer. For the Hele-Shaw-flow problem, a 5% numerical difference is found between the present method and the original theory. As a comparison, for the geometrical model a factor of about 100% was found by Hakim. Of course there is no *a priori* reason that a similar numerical factor should be obtained for the two problems. There is so far no numerical calculation of the prefactor for the Hele-Shaw-flow problem. We believe that such a calculation can offer insight to the quantitativeness of the solvability theory.

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